NASA-CR-198019

(NASA-CR-198019) ON THE TRELLIS STRUCTURE OF A (64,40,8) SUBCODE OF THE (64,42,8) THIRD-ORDER REED-MULLER CODE (Hawaii Univ.) 19 p

N95-24224

Unclas

G3/61 0045504

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Grant Number NAG 5-2613
Report Number 95-001

March 1, 1995

ON THE TRELLIS STRUCTURE OF A $(64, 40, 8)$ SUBCODE OF THE $(64, 42, 8)$ THIRD-ORDER REED-MULLER CODE

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Abstract

A $(64, 40, 8)$ subcode of the $(64, 42, 8)$ third-order Reed-Muller code is proposed to NASA for high-speed satellite communications. This code can be either used alone or used as an inner-code in a concatenated coding system with the NASA standard $(255, 223, 33)$ Reed-Solomon code as the outer code to achieve high performance with reduced decoding complexity. This Reed-Muller subcode has a relatively simple and parallel trellis structure and consequently can be decoded with a group of identical and relatively simple Viterbi decoders in parallel to achieve high-speed decoding. In this report, the complexities of various sectionalized trellis diagrams are analyzed. Based on this analysis, the trellis diagram with the smallest overall complexity will be used for the implementation of a high-speed decoder.

1. Introduction

A $(64, 40, 8)$ subcode of the third-order Reed-Muller (RM) code is proposed to NASA for high-speed satellite communications. This RM subcode can be used either alone or as an inner code of a concatenated coding system with the NASA standard $(255, 223, 33)$ Reed-Solomon (RS) code as the outer code to achieve high performance with reduced decoding complexity. It can also be used as a component code in a 3-level bandwidth efficient coded 8-PSK modulation system together with the $(64, 7, 32)$ first-order and $(64, 57, 4)$ fourth-order RM codes.

The $(64, 40, 8)$ RM subcode has a relatively simple and parallel trellis structure and consequently a group of identical and relatively simple Viterbi decoders can be devised to process the decoding in parallel. This not only simplifies the decoding complexity but also speeds up the decoding process. As a result, a high-speed decoder can be implemented. For the AWGN channel using BPSK transmission, this code achieves 5.3 dB coding gain over uncoded BPSK at the bit-error rate (BER) of 10^{-6} as shown in Figure 1. If this code is used as the inner code in a concatenated coding system with the NASA standard $(255, 223, 33)$ RS code as the outer code, the concatenated system will achieve error-free communication for signal-to-noise ratio (SNR) greater than 3 dB as shown in Figure 1. It achieves a 7.6 dB coding gain over the uncoded BPSK system at the BER of 10^{-6}. The $(64, 40, 8)$ RM subcode has a rate of 0.626 bits/symbol which is higher than that of the NASA standard rate-1/2 64-state convolutional inner code. In performance, the $(64, 40, 8)$ RM code achieves a 0.5 dB less coding gain than the NASA standard convolutional inner code. Because the $(64, 40, 8)$ RM subcode has a higher rate and can be implemented at a much higher decoding speed, it is an attractive alternative to replace the NASA standard convolutional inner code in a concatenated coding system with only a small loss in coding gain.

In this report, we investigate the complexities of various sectionalized trellis diagrams for the $(64, 40, 8)$ RM subcode. Particularily, we investigate the complexities of the 4-section, 8-section and 16-section trellis diagrams for the $(64, 40, 8)$ RM subcode. Based on this study, we will choose the trellis diagram which has the smallest overall complexity for decoder implementation.

2. Trellis Complexity of Binary Linear Block Codes

Any linear block code can be decoded by applying the Viterbi decoding algorithm to its minimal trellis using either Euclidean distance or correlation as the distance metric [1]. For BPSK transmission over an AWGN channel, the latter has distinct computational and implementation advantages. We define the computational complexity of any trellis as the number of addition equivalent operations needed to decode a single real valued received vector into the maximum likelihood solution (the codeword which has the highest a posteriori probability) by using the Viterbi algorithm based on the trellis. In a paper by Kasami et. al. [3], the computational complexity of a minimal L-section trellis of a block code has been calculated in terms of: (1) the number of real number comparisons, (2) the number of real number multiplications and (3) the number of real number additions. In the present case, by choosing to maximize the correlation between the received vector and the codewords we avoid the use of real number multiplications while decoding a received vector. Furthermore the complexity of a real number comparison is roughly the same as that of a real number addition. Hence we are justified in expressing the computational complexity of the various trellises in terms of the number of real number addition equivalent operations.

Definition: 1 *An L-section trellis diagram for a binary linear (N, K) block code* $\mathbf{C}$ *with $N = LM$ is a directed graph $\mathcal{T}$ with $L+1$ levels (henceforth called* depths*) labelled $0, M, 2M, \ldots, ML$ and vertex sets $S_0, S_M, \ldots, S_{ML}$ (henceforth called* state sets*) such that*

1. *$S_0 = \{\rho_o\}$, $S_N = S_{LM} = \{\rho_F\}$ where ρ_o and ρ_F denote the initial and final states respectively.*

2. *There are edges (henceforth called* branches*) connecting states in adjacent state sets $S_{iM}, S_{(i+1)M}$, $0 \leq i < L$ with each branch labelled by a binary M-tuple and originating at a state in S_{iM}.*

3. *There is a directed path from the initial state ρ_o to the final state ρ_F with label sequence $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_L$ if and only if*
$$(\mathbf{u}_1 \circ \mathbf{u}_2 \circ \cdots \circ \mathbf{u}_L)$$
is a codeword in $\mathbf{C}$ *where $\circ$ stands for concatenation of two sequences.*

Let $\mathbf{C}$ denote an $(N, K, d_{\min})$ linear block code. Let L, M be positive integers such that $LM = N$. Given a L-section trellis, the set of states at the end of each section

$\{S_0, S_M, S_{2M}, \ldots, S_{(L-1)M}, S_{LM}\}$, we define a sequence $\{s_0, s_M, \ldots, s_{LM}\}$ called the **state dimension profile** (SDP) of the trellis and given by $s_{iM} = \log_2(|S_{iM}|)$. The minimal L-section trellis of a code $\mathbf{C}$ has the property that every component of its SDP is less than or equal to the corresponding component in the SDP of any other proper L-section trellis for $\mathbf{C}$. The maximum among the $N+1$ components in the SDP of the minimal N-section trellis for $\mathbf{C}$ is denoted $s_{\max}(\mathbf{C})$ and we will denote the maximum of the components in the SDP of the minimal L-section trellis for $\mathbf{C}$ as $s_{\max,L}(\mathbf{C})$. For a binary N-tuple $\mathbf{v} = (v_1, \ldots, v_N)$, let $p_{h,h'}[\mathbf{v}]$ denote the $(h'-h)$-tuple $(v_{h+1}, \ldots, v_{h'})$ and let

$$p_{h,h'}[\mathbf{C}] = \{p_{h,h'}[\mathbf{c}] : \ \mathbf{c} \in \mathbf{C}\}$$

Let $\mathbf{C}_{h,h'}$ be the linear subcode of $\mathbf{C}$ consisting of all codewords whose components are all zero except for the $(h'-h)$ components from the $(h+1)$-th bit position to the h'-th bit position.

The computational complexity of any trellis $\mathcal{T}$ is denoted $A(\mathcal{T})$ and is composed of two components, one as a result of branch metric computations denoted $A_B(\mathcal{T})$ and the other due to the state metric computations and comparisons denoted $A_L(\mathcal{T})$ yielding $A(\mathcal{T}) = A_B(\mathcal{T}) + A_L(\mathcal{T})$.

In a L-section minimal trellis for a block code, there may be a set of parallel branches between two adjacent states. In such a case, we call the entire set of parallel branches a **composite branch**. Each composite branch in the i-th section $1 \leq i \leq L$, is made up of 2^{P_i} parallel branches where P_i is the dimension of the subcode denoted $\mathbf{C}^{tr}_{(i-1)M,iM}$ as shown in [2]. In the i-th section of a L-section trellis for a linear block code $1 \leq i \leq L$, the number of distinct branch metrics that have to be computed is 2^{D_i} where D_i is the dimension of the subcode $p_{(i-1)M,iM}(\mathbf{C})$ and this number is much less than the total number of branches. D_i is the rank of the submatrix formed by M columns from the $((i-1)M+1)$-th to the (iM)-th column of the generator matrix of the code and is upper bounded by M. For $1 \leq i \leq L$, let the number of composite branches merging into any state $s \in S_{iM}$ be $2^{\delta_{iM}}$ (it is the same for any state in S_{iM}). For $0 \leq i < L$, let the number of composite branches emanating from any state $s \in S_{iM}$ be $2^{\lambda_{iM}}$, (it is the same for any state in S_{iM}). δ_{iM} is related to $\lambda_{(i-1)M}$ as follows:

$$\delta_{iM} = s_{(i-1)M} + \lambda_{(i-1)M} - s_{iM}$$

Therefore the $\{\delta_{iM} : \ 1 \leq i \leq L\}$ can be computed from the SDP of the code and $\{\lambda_{iM} : 0 \leq i < L\}$. Based on the theory of L-section trellises [2, 4] it can be shown that

$$\lambda_{(i-1)M} = \dim(\mathbf{C}_{(i-1)M,N}) - \dim(\mathbf{C}_{iM,N}) - P_i$$

These dimensions can be easily determined from the trellis oriented generator matrix of the code [4].

For the special case of Reed-Muller codes and $L = 2^l$ for a positive integer l, since the L-section minimal trellis is symmetric about the midpoint [2], i.e., the last $L/2$ sections form a mirror image of the first $L/2$ sections and we have the following.

$$s_{iM} = s_{(L-i)M},\ 0 \le i \le L/2, \text{ and } \lambda_{iM} = \delta_{(L-i)M},\ 0 \le i < L$$

The five parameters $s_{iM}, D_{iM}, P_{iM}, \lambda_{iM}, \delta_{iM}$ discussed above are essential to calculating the computational complexity of a minimal trellis.

Consider the operation of a Viterbi decoder using a L-section minimal trellis of a linear block code. In order to compute the state metric of a state $s \in S_{iM}$, $1 \le i \le L$, the decoder has to pick the minimum among $2^{\delta_{iM}}$ candidates. For $1 \le j \le 2^{\delta_{iM}}$, the j-th candidate metric is given by the sum of the metric of the j-th composite branch and the metric of the originating state of the j-th composite branch. Thus are $2^{\delta_{iM}}$ additions and $2^{\delta_{iM}} - 1$ comparisons are required to compute the state metric of s. Hence we have the number of addition equivalent operations as

$$A_L(\mathcal{T}) = \sum_{i=1}^{L} 2^{s_i} \left(2^{\delta_{iM}+1} - 1\right) \tag{2.1}$$

In the i-th section of the trellis, the decoder has to compute the 2^{D_i} distinct simple branch metrics at the cost of $(M-1)$ additions per branch metric. If the branch metric computations are done in the most parallel manner with 2^{D_i} adders operating simultaneously in parallel and independent of each other, a total of $2^{D_i}(M-1)$ additions is required. There are other slower methods of computing the same set of branch metrics that use lesser number of additions. In addition to the distinct simple branch metrics, the branch metric of each composite branch needs to be computed. The branch metric of a composite branch is the minimum of the branch metrics of the 2^{P_i} simple branches that form a composite branch. Given the branch metrics of all the simple branches, $2^{P_i} - 1$ comparisons are required to compute the metric of a composite branch. Therefore, the number of additions required to compute the metric of all composite branches is given by the number of distinct composite branches times $(2^{P_i} - 1)$. Based on the above discussion we have,

$$A_B(\mathcal{T}) = \sum_{i=1}^{L} 2^{D_i}(M-1) + \sum_{i=1}^{L} \#(\text{Distinct composite branches in section-}i)\ (2^{P_i} - 1) \tag{2.2}$$

Based on the above discussion, a study of all the possible L-section trellises for the $(32, 16, 8)$ Reed-Muller code yielded Table 1.

From Table 1, it is evident that the computational complexity as well as the maximum state space dimension depend on the number of sections in the trellis of the code. For the above code, although $A(\mathcal{T})$ reaches a minimum of 4,064 when $L = 8$, the trellis with $L = 4$ and $A(\mathcal{T}) = 6,847$ is more suitable for implementation because while the former has only 2 parallel isomorphic trellises with each subtrellis having $s_{\max} = 128$ states the latter has 8 parallel isomorphic subtrellises with each subtrellis having $s_{\max} = 8$ states.

3. Trellises with Parallel Structure

The objective of this section is to show that we can build a trellis for a linear block code say C which is a union of certain desired number of parallel isomorphic subtrellises. *Although this trellis is not minimal, it has a maximum state space dimension less than or equal to* $s_{\max}(\mathrm{C})$. The conditions under which such a trellis construction is possible and an upper bound on the number of such parallel subtrellises is derived. In some cases the minimal trellis itself possesses a parallel structure. The number of such parallel subtrellises (if any) in the minimal trellis is derived.

Suppose we want to build a trellis for linear block code C as a union of 64 parallel isomorphic subtrellises. This can be done by first choosing a subcode C$'$ of dimension 6 less than the dimension of C. Second, we build the trellis for C$'$. Finally, the trellis for C can be obtained as the union of 64 parallel isomorphic subtrellises with each subtrellis being isomorphic to the trellis of C$'$. The resulting trellis for C may not be minimal. We prove necessary and sufficient conditions for the existence of a subcode C$'$ of C such that the resulting trellis for C has maximum state space dimension $s_{\max}(\mathrm{C})$ while retaining the property of parallelism.

Trellises with parallel structure are desirable for the VLSI implementation of high speed decoders for block codes. One can envision a decoder consisting of as many parallel processors as there are parallel subtrellises in the total trellis, each processor producing a best candidate from its subtrellis after processing it with the Viterbi algorithm. Finally it is a simple matter of choosing the best codeword among the choices presented by the processors.

Parallel Trellises with Constraint on Maximum State Space Dimension

Given a linear block code, several non-isomorphic trellises are possible. The number of such non-isomorphic trellises is related to the placement of non-attacking rooks on an upper triangular chess board [5]. However there is a unique proper trellis called the minimal trellis such that its state space dimension is less than or equal to the state space dimension of any other proper trellis at every position. A systematic method for the construction of the minimal trellis using the *trellis oriented generator* matrix has been presented in [4]. One drawback of the minimal trellis is that often it does not have any parallel structure; or the parallel subtrellises are themselves too large. In such a case it is desirable to explore the possibility of increasing the parallelism of the trellis even if minimality has to be compromised. A trellis construction with large number of parallel subtrellises for the special case of Reed-Muller codes has been presented in [6]. This so called *primitive decomposition* has the drawback that the state space dimension is very large and usually much larger than the maximum for the minimal trellis. For many codes the primitive decomposition is quite attractive.

Let $\mathbf{G}$ be the trellis oriented generator matrix of an (N, K) binary block code $\mathbf{C}$. Let $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ be a typical row of $\mathbf{G}$. Then, we define **the span of r** denoted $span(\mathbf{r})$ to be the smallest interval $[i, j]$, $1 \leq i \leq j \leq N$ which contains all the non-zero elements of $\mathbf{r}$. For a row $\mathbf{r}$ whose span is $[i, j]$ we also define an **active span of r** denoted $aspan(\mathbf{r})$ as $[i, j-1]$ if $i < j$ and $aspan(\mathbf{r}) = \phi$ if $i = j$. The trellis oriented matrix has the following properties: 1) The leading 1 of every row occurs in an earlier position than the leading 1 of the row below it. 2) The trailing 1 of every row occurs at a different position from the trailing 1 of every other row. 3) Any other trellis oriented matrix for $\mathbf{C}$ has the same set of **row spans** although the rows themselves may be different [5]. Let $\mathcal{T}$ be the minimal n-section trellis for $\mathbf{C}$. It can be proved that given the trellis oriented generator matrix of a code, the state space dimension at any position l is just equal to the number of rows whose active span contain l [5, 4]. For example, consider the following trellis oriented generator matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & r_1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & r_2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & r_3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & r_4 \end{pmatrix}$$

for which $aspan(\mathbf{r}_1) = [1, 3]$, $aspan(\mathbf{r}_2) = [2, 6]$, $aspan(\mathbf{r}_3) = [3, 5]$ and $aspan(\mathbf{r}_4) = [5, 7]$. For each l, $0 \leq l \leq 8$ counting the number of rows which are active at each l for $0 \leq l \leq 8$ yields the state dimension profile (SDP) $\{0, 1, 2, 3, 2, 3, 2, 1, 0\}$. For $0 \leq l \leq N$, let $s_l(\mathbf{C})$ denote the dimension of the l-th state space of $\mathbf{C}$. Let $s_{\max}(\mathbf{C})$ be the maximum among the state space dimensions. Define the non-empty set

$$I_{\max}(\mathbf{C}) = \{l : s_l(\mathbf{C}) = s_{\max}(\mathbf{C})\} \tag{3.3}$$

Suppose we choose a subcode $\mathbf{C}'$ of $\mathbf{C}$ such that $\dim(\mathbf{C}') = \dim(\mathbf{C}) - 1$ and the set of coset representatives $[\mathbf{C}/\mathbf{C}']$ is generated by the single row $\mathbf{r} \in \mathbf{G}$. From the above statement about $s_l(\mathbf{C})$, it is clear that $s_l(\mathbf{C}') = s_l(\mathbf{C}) - 1$ for exactly those l where $\mathbf{r}$ is active. i.e., $l \in aspan(\mathbf{r})$. For other positions $l \notin aspan(\mathbf{r})$ we have $s_l(\mathbf{C}') = s_l(\mathbf{C})$. Hence we have the following proposition.

Proposition: 1 *If there exists a row* $\mathbf{r}$ *in the trellis oriented generator matrix* $\mathbf{G}$ *for the code* $\mathbf{C}$ *such that* $aspan(\mathbf{r}) \supseteq I_{\max}(\mathbf{C})$ *then, we can form a subcode* $\mathbf{C}'$ *of* $\mathbf{C}$ *generated by* $\mathbf{G} - \{\mathbf{r}\}$ *such that* $s_{\max}(\mathbf{C}') = s_{\max}(\mathbf{C}) - 1$ *and* $I_{\max}(\mathbf{C}') \supseteq I_{\max}(\mathbf{C})$.

In fact $I_{\max}(\mathbf{C}') = I_{\max}(\mathbf{C}) \cup \{l : s_l(\mathbf{C}) = s_{\max}(\mathbf{C}) - 1, l \notin aspan(\mathbf{r})\}$. Since $\mathbf{G}$ is a trellis oriented generator matrix, $\mathbf{G}' = \mathbf{G} - \{\mathbf{r}\}$ is also trellis oriented. We can apply the above proposition again to $\mathbf{C}'$ if there exists another row $\mathbf{r}' \in \mathbf{G}'$ with $aspan(\mathbf{r}') \supseteq I_{\max}(\mathbf{C}')$. This yields a subcode $\tilde{\mathbf{C}}$ with dimension smaller by one and $s_{\max}(\tilde{\mathbf{C}}) = s_{\max}(\mathbf{C}') - 1$. If no such row $\mathbf{r}'$ exists the proposition cannot be applied and the recursion stops. The above proposition can be generalized.

Let $R(\mathbf{C})$ be the subset of rows of $\mathbf{G}$, the trellis oriented generator matrix of a code $\mathbf{C}$ consisting of those rows $\mathbf{r}$ given by

$$R(\mathbf{C}) = \{\mathbf{r} \in \mathbf{G} : aspan(\mathbf{r}) \supseteq I_{\max}(\mathbf{C})\} \tag{3.4}$$

Let $d = |R(\mathbf{C})|$ where $|Q|$ denotes the cardinality of any finite set Q.

Proposition: 2 *With* $R(\mathbf{C})$ *defined as above and* $d = |R(\mathbf{C})|$, *let* $1 \leq d' \leq d$. *There exists a subcode* $\mathbf{C}'$ *of* $\mathbf{C}$ *such that* $s_{\max}(\mathbf{C}') = s_{\max}(\mathbf{C}) - d'$ *and* $\dim(\mathbf{C}') = \dim(\mathbf{C}) - d'$ *if and only if there exists a subset* $R' \subseteq R(\mathbf{C})$ *consisting of* d' *rows of* $R(\mathbf{C})$ *such that for every* l *satisfying* $s_l(\mathbf{C}) > s_{\max}(\mathbf{C}')$ *there exist at least* $s_l(\mathbf{C}) - s_{\max}(\mathbf{C}')$ *rows in* R' *whose active span contain* l. *The set of coset representatives* $[\mathbf{C}/\mathbf{C}']$ *is generated by* R'.

Proof: Suppose $R' = \{\mathbf{r}'_1, \ldots, \mathbf{r}'_{d'}\}$ satisfies the conditions in the hypothesis. Since $R' \subseteq R(\mathbf{C})$, $I_{\max}(\mathbf{C}) \subseteq aspan(\mathbf{r}'_i)$ for $1 \leq i \leq d'$. Consider the subcode generated by $\mathbf{G} - R'$. For those $l \in I_{\max}(\mathbf{C})$, we can determine $s_l(\mathbf{C}')$ by counting the number of rows $\mathbf{r} \in (\mathbf{G} - R')$ that are active at that position l. But this number is exactly less than $s_{\max}(\mathbf{C})$ by d'. For $l \notin I_{\max}(\mathbf{C})$ and satisfying $s_l(\mathbf{C}) > s_{\max}(\mathbf{C}')$, we are assured by the hypothesis that $s_l(\mathbf{C})$ will be reduced by at least $s_l(\mathbf{C}) - s_{\max}(\mathbf{C}')$ thus guaranteeing that $s_{\max}(\mathbf{C}') = s_{\max}(\mathbf{C}) - d'$.

To prove the converse, let $\mathbf{C}'$ be a subcode of $\mathbf{C}$ whose dimension is $\dim(\mathbf{C}) - d'$ and satisfying $s_{\max}(\mathbf{C}') = s_{\max}(\mathbf{C}) - d'$. Without loss of generality we may let $\mathbf{C}'$ be generated $\mathbf{G} - R'$ for some subset R' of the trellis oriented generator matrix $\mathbf{G}$ of $\mathbf{C}$ with $|R'| = d'$. Let $\tilde{\mathcal{T}}$ be the minimal trellis corresponding to $\mathbf{G}$. Let $\mathcal{T}'$ be the minimal trellis for $\mathbf{C}'$. Let $N_l(R')$ be the number of rows $\mathbf{r}'$ in R' such that $l \in aspan(\mathbf{r}')$. Then, at every position l, $0 \leq l \leq N$, we have

$$s_l(\tilde{\mathcal{T}}) = s_l(\mathbf{C}') + N_l(R') \geq s_l(\mathbf{C}) \tag{3.5}$$

since $s_l(\mathbf{C})$ is the smallest possible state space dimension. Therefore

$$\begin{aligned} N_l(R') &\geq s_l(\mathbf{C}) - s_l(\mathbf{C}') \\ N_l(R') &\geq s_l(\mathbf{C}) - s_{\max}(\mathbf{C}') \end{aligned} \tag{3.6}$$

For every l at least $s_l(\mathbf{C}) - s_{\max}(\mathbf{C}')$ rows of R' are active. Also, for every $l \in I_{\max}(\mathbf{C})$ we have $N_l(R') \geq s_{\max}(\mathbf{C}) - s_{\max}(\mathbf{C}') = d'$. So all the rows $\mathbf{r}' \in R'$ satisfy $aspan(\mathbf{r}') \supseteq I_{\max}(\mathbf{C})$. Thus $R' \subseteq R(\mathbf{C})$. ∎

The utility of the above proposition is that it shows how to choose a subcode $\mathbf{C}'$ of $\mathbf{C}$ with $s_{\max}(\mathbf{C}') = s_{\max}(\mathbf{C}) - \dim([\mathbf{C}/\mathbf{C}'])$, such that one can build a trellis $\mathcal{T}$ for $\mathbf{C}$ which although not minimal has the following properties.

1. The maximum state space dimension of $\mathcal{T}$ is $s_{\max}(\mathbf{C})$.
2. $\mathcal{T}$ is the union of $2^{\dim\left[\mathbf{C}/\mathbf{C}'\right]}$ parallel isomorphic subtrellises $\mathcal{T}_i$ with each $\mathcal{T}_i$ being isomorphic to the minimal trellis for $\mathbf{C}'$.
3. The smallest such subcode has dimension lower bounded by $\dim(\mathbf{C}) - |R(\mathbf{C})|$. i.e., the maximum number of parallel subtrellises one can obtain with the constraint that the total state space dimension never exceed $s_{\max}(\mathbf{C})$ is upper bounded by $2^{|R(\mathbf{C})|}$ with $R(\mathbf{C})$ as defined above.

The following proposition follows from the above discussion.

Proposition: 3 *The logarithm to the base 2 of the number of parallel isomorphic subtrellises in a minimal L-section trellis for a binary (n, k) linear block code is given by the number of rows in its trellis oriented generator matrix whose span contain the integers $\{M, 2M, \ldots, (L-1)M\}$ where $N = LM$.*

As an example consider the extended code $BCH(32, 21, 6)$ for which the minimal 4-section trellis has SDP $\{0, 7, 9, 7, 0\}$. Any trellis oriented generator matrix for this code has the following unique set of row spans.

$$\begin{aligned} span(\mathbf{r}_1) &= [1, 8] \quad span(\mathbf{r}_2) = [2, 15] \\ span(\mathbf{r}_3) &= [3, 13] \quad span(\mathbf{r}_4) = [4, 14] \\ span(\mathbf{r}_5) &= [5, 12] \quad span(\mathbf{r}_6) = [6, 18] \\ span(\mathbf{r}_7) &= [7, 21] \quad span(\mathbf{r}_8) = [8, 25] \end{aligned}$$

$$
\begin{aligned}
span(\mathrm{r}_9) &= [9,16] \quad span(\mathrm{r}_{10}) = [10,23] \\
span(\mathrm{r}_{11}) &= [11,19] \quad span(\mathrm{r}_{12}) = [12,26] \\
span(\mathrm{r}_{13}) &= [13,20] \quad span(\mathrm{r}_{14}) = [14,22] \\
span(\mathrm{r}_{15}) &= [15,27] \quad span(\mathrm{r}_{16}) = [17,24] \\
span(\mathrm{r}_{17}) &= [18,31] \quad span(\mathrm{r}_{18}) = [19,29] \\
span(\mathrm{r}_{19}) &= [20,30] \quad span(\mathrm{r}_{20}) = [21,28] \\
span(\mathrm{r}_{21}) &= [25,32]
\end{aligned}
$$

$I_{\mathbf{max}}(\mathrm{C}) = \{16\}$ and it can be verified that $|R(\mathrm{C})| = 9$. In an attempt to build a trellis consisting of 64 parallel subtrellises while satisfying the upper bound of 9 on the maximum state space dimension, we let $d' = 6$. So $s_{\mathbf{max}}(\mathrm{C}) - d' = s_{\mathbf{max}}(\mathrm{C}') = 3$. The set $\{l : s_l(\mathrm{C}) > s_{\mathbf{max}}(\mathrm{C}')\} = \{8, 16, 24\}$. However, we find that no subset R' of $R(\mathrm{C})$ exists satisfying the conditions in proposition 2. Hence we cannot build a trellis consisting of 64 parallel subtrellises for this code without violating the constraint on the maximum state space dimension. If we choose $d' = 5$ then we can find a subset $R' = \{\mathrm{r}_6, \mathrm{r}_7, \mathrm{r}_8, \mathrm{r}_{12}, \mathrm{r}_{15}\} \subseteq R(\mathrm{C})$ that satisfies all the conditions in proposition 2. Hence choosing the subcode C' generated by $G - R'$ we obtain a trellis $\mathcal{T}$ for C consisting of 32 parallel isomorphic subtrellises. Each subtrellis is isomorphic to the minimal trellis for C' which has $s_{\mathbf{max}}(\mathrm{C}') = 4$.

4. The Trellis with Least Complexity for the (64, 40, 8) Code

Let $\mathbf{C}$ denote the RM (64, 42, 8) code and $\tilde{\mathrm{C}}$ a (64, 40) subcode of $\mathbf{C}$. We want to examine all possible L-section trellises with an aim to find their suitability for decoding. We realize that if the trellis to be used for decoding purposes has a parallel structure and is composed of a union of M parallel subtrellises, then M smaller Viterbi decoders can be designed to process each subtrellis simultaneously in parallel. Therefore the **effective complexity** is just that of a single subtrellis plus the cost of obtaining the final decision from each of the M Viterbi decoders. We determine the value of L which minimizes the effective complexity with the constraint that the L-section trellis $\tilde{\mathcal{T}}$ have a maximum state complexity not greater than $s_{\mathbf{max},L}(\tilde{\mathrm{C}})$. This constraint is reasonable because although one can essentially increase the number of parallel subtrellises without bound the hardware complexity of implementing the resulting trellis and the cost of combining outputs from subtrellis decoders both increase exponentially in the state-space dimension of the code.

Thus, we will determine the smallest effective complexity denoted $A_{eff}(L)$, that can be achieved with an L-section trellis satisfying the constraint on maximum state dimension among the choices $L = 1, 2, 4, 8, 16, 32, 64$ and pick that L which yields the minimum $A_{eff}(L)$. The uninteresting choices of $L = 1, 2, 32, 64$ were eliminated from consideration by analysis which is not presented here. Herein, we consider consider the more interesting cases of $L = 4, 8$ and 16.

4.1. $L = 4, M = 16$

Let $\mathrm{C}_0 = (16, 15, 2)$, $\mathrm{C}_1 = (16, 11, 4)$ $\mathrm{C}_2 = (16, 5, 8)$ be the corresponding Reed-Muller codes, G_i a generator matrix of C_i and $\mathrm{G}_{i/j}$ a generator matrix for the set of coset representatives $[\mathrm{C}_i/\mathrm{C}_j]$. For $L = 4$ the RM $(64, 42)$ code has a minimal trellis (with 16 parallel subtrellises) corresponding to the 2-level squaring construction with a state dimension profile (SDP) $\{0, 10, 10, 10, 0\}$ ($s_{\max,4}(\mathrm{C}) = 10$) and trellis oriented generator matrix

$$
\begin{aligned}
\mathrm{G} = \; & \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \otimes \mathrm{G}_{0/1} + \\
& \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \otimes \mathrm{G}_{1/2} + \\
& \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \mathrm{G}_2
\end{aligned}
\tag{4.7}
$$

In order to obtain a generator matrix for $\tilde{\mathrm{C}}$, the $(64, 40)$ subcode of C, one can delete any two of the 64 rows above. The $s_{\max,4}(\tilde{\mathrm{C}})$ of the resulting code depends on which two rows we delete. It is easy to see that in order to have the least $s_{\max}(\tilde{\mathrm{C}}, 4)$ which equals 8 we must delete any two of the 4 rows among $(1111) \otimes \mathrm{G}_{0/1}$ obtaining an SDP of $\{0, 8, 8, 8, 0\}$ ($s_{\max,4}(\tilde{\mathrm{C}}) = 4$). Thus $I_{\max}(\tilde{\mathrm{C}}) = \{1, 2, 3\}$ and $R(\tilde{\mathrm{C}}) = \{(1111) \otimes {\mathrm{G}'}_{0/1}\}$ where ${\mathrm{G}'}_{0/1}$ has dimension 2. Since $|R(\tilde{\mathrm{C}})| = 2$, we can obtain at most 4 parallel subtrellises in any 4-section without exceeding the allowable $s_{\max,4}$ of 8. Choosing C' as the subcode of $\tilde{\mathrm{C}}$ generated by

$$
\begin{aligned}
\mathrm{G}' = \; & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \otimes \mathrm{G}_{1/2} + \\
& \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \mathrm{G}_2
\end{aligned}
\tag{4.8}
$$

we can obtain a trellis for $\tilde{C}$ as the union of 4 parallel isomorphic subtrellises $\mathcal{T}'_i$, $i = 0,1,2,3$. The complexity parameters of the subtrellises is presented in Table 2. The resulting computational complexity parameters are

$$A_B(\mathcal{T}'_i) = 23,296 \text{ and } A_L(\mathcal{T}'_i) = 16,383 \tag{4.9}$$

This gives $A_{eff}(4) = 23,296 + 16,383 + 3 = 39,682$ addition equivalent operations.

4.2. $L = 8, M = 8$

Let $C_0 = (8,8,1)$, $C_1 = (8,7,2)$ $C_2 = (8,4,4)$ $C_3 = (8,1,8)$ be the corresponding Reed-Muller codes. For $L = 8$, the RM $(64,42,8)$ code has a minimal trellis (with 2 parallel subtrellises) corresponding to the 3-level squaring construction with a SDP $\{0,7,10,13,10,13,10,7,0\}$ ($s_{\max,8}(C) = 13$) with trellis oriented generator matrix

$$\acute{G} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & r_0^0 \end{pmatrix} \otimes G_{0/1} + \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & r_0^1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & r_1^1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & r_2^1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & r_3^1 \end{pmatrix} \otimes G_{1/2} + \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \otimes G_{2/3} + I_8 \otimes G_3 \tag{4.10}$$

The $(64,40)$ subcode $\tilde{C}$ with the best SDP is obtained by deleting the rows $r_0^0 \otimes G_{0/1}$ and any one among the three rows $r_1^1 \otimes G_{1/2}$. This code $\tilde{C}$ has SDP $\{0,6,8,11,8,11,8,6,0\}$ ($s_{\max,8}(\tilde{C}) = 11$) and thus $I_{\max}(\tilde{C}) = \{3,5\}$. By inspection $R(\tilde{C}) = \{r_2^1 \otimes G_{1/2}, r_1^1 \otimes G'_{1/2}\}$ yielding $|R(\tilde{C})| = 5$. Thus one can obtain at most 32 parallel subtrellises in any 8-section trellis for $\tilde{C}$ without exceeding the maximum allowable state space dimension of $s_{\max,8}(\tilde{C}) = 11$. Let C' denote the $(64,35)$ subcode of $\tilde{C}$ generated by $G' = \tilde{G} - R(\tilde{C})$ with SDP $\{0,6,6,6,3,6,6,6,0\}$.

Hence one can build a trellis for $\tilde{C}$ as the union of 32 parallel isomorphic subtrellises $\mathcal{T}'_i$, $0 \leq i < 32$. The complexity parameters of the subtrellises is presented in Table 3. The resulting computational complexity parameters are

$$A_B(\mathcal{T'}_i) = 6,784, \quad \text{and} \quad A_L(\mathcal{T'}_i) = 6,007 \tag{4.11}$$

This gives $A_{eff}(8) = 6,784 + 6,007 + 31 = 12,822$ addition equivalent operations.

4.3. L = 16, M = 4

Let $C_0 = C_1 = (4,4,1)$, $C_2 = (4,3,2)$ $C_3 = (4,1,4)$ $C_4 = (4,0,\infty)$ be the corresponding Reed-Muller codes. For $L = 16$, the RM $(64,42,8)$ code has a trellis oriented generator matrix given by

$$G = G_{RM(16,5,8)} \otimes G_{1/2} + G_{RM(16,11,4)} \otimes G_{2/3} + G_{RM(16,15,2)} \otimes G_{3/4}. \tag{4.12}$$

where $G_{RM(n,k,d)}$ denotes a trellis oriented generator matrix for the corresponding Reed-Muller code. For $L = 16$, the RM $(64,42,8)$ code has a minimal trellis (with no parallel subtrellises) corresponding to the 4-level squaring construction with a SDP $\{0,4,7,10,10,13,13,13,10,13,13,13,10,10,7,4,0\}$ ($s_{max,16}(C) = 13$). The $(64,40)$ subcode $\tilde{C}$ with the best SDP is generated by $\tilde{G} = G - \{r_1^1 \otimes G_{1/2}, r_2^1 \otimes G_{1/2}\}$ where r_1^1 and r_2^1 are the two rows with span $[2,15]$ and $[3,14]$ in the trellis oriented generator matrix for RM$(16,11,4)$. The SDP of the minimal trellis for $\tilde{C}$ is $\{0,4,6,8,8,11,11,11,8,11,11,11,8,8,6,4,0\}$ ($s_{max,16}(\tilde{C}) = 11$). $I_{max}(\tilde{C}) = \{5,6,7,9,10,11\}$ and $R(\tilde{C}) = \{r_3^1 \otimes G_{1/2}, r_3^2 \otimes G_{2/3}\}$ where r_3^1 is the row with span $[5,12]$ in the trellis oriented generator matrix for RM$(16,5,8)$ and r_3^2 is the row with span $[4,12]$ in the trellis oriented generator matrix for RM$(16,11,4)$. Since $|R(\tilde{C})| = 3$, one can obtain at most 8 parallel subtrellises in any 16-section trellis for $\tilde{C}$ without exceeding the allowable $s_{max,16}(\tilde{C})$ of 11. Choosing C' as the subcode generated by $G' = \tilde{G} - R(\tilde{C})$, we obtain a subcode having the SDP $\{0,4,6,8,6,8,8,8,5,8,8,8,6,8,6,4,0\}$. Hence we can obtain a trellis for $\tilde{C}$ as the union of 8 parallel isomorphic subtrellises $\mathcal{T'}_i$, $0 \leq i < 8$. The complexity parameters of the subtrellises is presented in Table 4. The resulting computational complexity parameters are

$$A_B(\mathcal{T'}_i) = 384, \quad \text{and} \quad A_L(\mathcal{T'}_i) = 22,783 \tag{4.13}$$

This gives $A_{eff}(16) = 384 + 22,783 + 7 = 23,174$ addition equivalent operations.

4.4. L = 32, M = 2 and L = 64, M = 1

When L = 32, $s_{max,32}(\tilde{C}) = 12$. The maximum number of parallel isomorphic subtrellises possible without exceeding the allowable $s_{max,32}(\tilde{C}) = 12$ in any 32-section trellis for the $(64,40)$ subcode $\tilde{C}$ is at most 4. So $A_{eff}(32) \geq 37,476$. When L = 64,

$s_{\mathbf{max},64}(\tilde{\mathrm{C}}) = 12$. Furthermore, no parallel subtrellises are possible without exceeding the allowable $s_{\mathbf{max},64}(\tilde{\mathrm{C}}) = 12$. Hence $A_{eff}(64) = 198,000$.

4.5. Conclusion

By comparing the $A_{eff}(L)$ obtained for $L = 1, 2, 4, 8, 16, 32$ and 64 we find that the **non-minimal** 8-section trellis has the least effective decoding computational complexity for the $(64, 40)$ subcode of the Reed-Muller $(64, 42, 8)$ code. This trellis has a parallel structure with 32 parallel isomorphic subtrellises with each subtrellis having at most 64 states.

Table-1: Decoding Complexity for the RM(32,16,8) Code

L	$A(\mathcal{T})$	$s_{\max,L}(C)$
1	1,048,560	1
2	65,599	64
4	6,847	64
8	4,064	256
16	5,159	256
32	7,995	512

Table 2: Complexity Parameters of 4-section trellis for $(64, 38)$ code

i	0	1	2	3	4
$\dim(S_{iM})$	0	6	6	6	0
$\dim(D_i)$	-	10	10	10	10
$\dim(P_i)$	-	5	5	5	5
$\dim(\delta_i)$	-	0	6	6	6
$\dim(\lambda_i)$	6	6	6	0	-

Table 3: Complexity Parameters of 8-section trellis for $(64, 35)$ code

i	0	1	2	3	4	5	6	7	8
$\dim(S_{iM})$	0	6	6	6	3	6	6	6	0
$\dim(D_i)$	-	6	6	6	6	6	6	6	6
$\dim(P_i)$	-	1	1	1	1	1	1	1	1
$\dim(\delta_i)$	-	0	3	3	6	3	3	3	6
$\dim(\lambda_i)$	6	3	3	3	6	3	3	0	-

Table 4: Complexity Parameters of 16-section trellis for $(64, 37)$ code

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\dim(S_{iM})$	0	4	6	8	6	8	8	8	5	8	8	8	6	8	6	4	0
$\dim(D_i)$	-	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
$\dim(P_i)$	-	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\dim(\delta_i)$	-	0	1	1	3	1	3	3	4	1	3	3	3	1	3	3	4
$\dim(\lambda_i)$	4	3	3	1	3	3	3	1	4	3	3	1	3	1	1	0	-